

Summary: Vector operations so far $c \mathbf{v}$ = parallel to $\mathbf{v}$, scaled by c $\mathbf{a} + \mathbf{b}$ = vector sum / resultant force**Entry Task** (HW #2–4)A. $\mathbf{a} = \langle 3, 1, 2 \rangle$ $\mathbf{b} = -\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ Compute $\mathbf{a} \cdot \mathbf{b}$.**12.3 Dot Products (new)**If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,
define

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product gives **a number (scalar)**.B. For three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, what type
of object would each of the following give?*One is a “trick” question.*

- $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

Basic Fact List (HW #1–4)

Manipulation facts (like multiplication):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$(c \mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c \mathbf{b})$$

$$\mathbf{a} \cdot \mathbf{0} = 0$$

Helpful side fact:

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

Discussion:

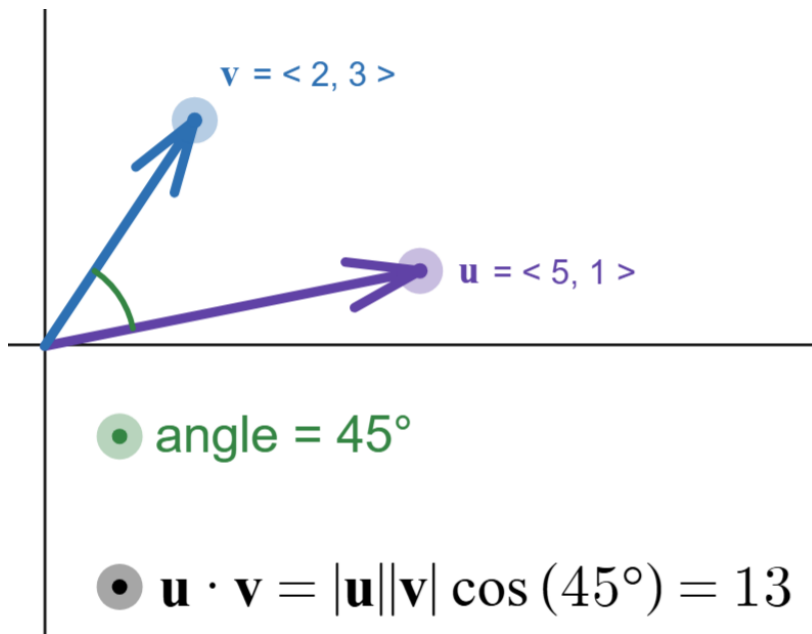
How would you *prove* these facts?

The Most Important Fact (HW 5–8)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

θ is the angle between the vectors when they are drawn TAIL-TO-TAIL.

Solving for the angle: $\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$

**Why is the formula true?**

Fact 1: The law of cosines gives:

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

Fact 2: The dot product rules give:

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2. \end{aligned}$$

Put these two facts together and simplify!

Something to think about

- $\mathbf{a} \cdot \mathbf{b} > 0 \Rightarrow$ what kind of angle?
- $\mathbf{a} \cdot \mathbf{b} < 0 \Rightarrow$ what kind of angle?
- $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow$ what kind of angle?

Most important consequence:

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

Orthogonal, Parallel, or Neither? (HW #8–9)

Orthogonal: $\mathbf{a} \cdot \mathbf{b} = 0$

Parallel: $\mathbf{a} = c \mathbf{b}$ for some scalar c

Otherwise: neither.

Angles of Intersection (HW 10 and 11)

Convert each slope into a direction vector,
then use the dot product.

Example: Angle between lines.

Find the acute angle of intersection of

$$y = \frac{3}{4}x + 4 \text{ and } y = -\frac{1}{2}x + 10$$

Example: Angle between other curves.

Find the angle of intersection of

$$y = x^2 \text{ and } y = 2x + 3 \text{ at the point } (3,9)$$

Scalar / Component Projection

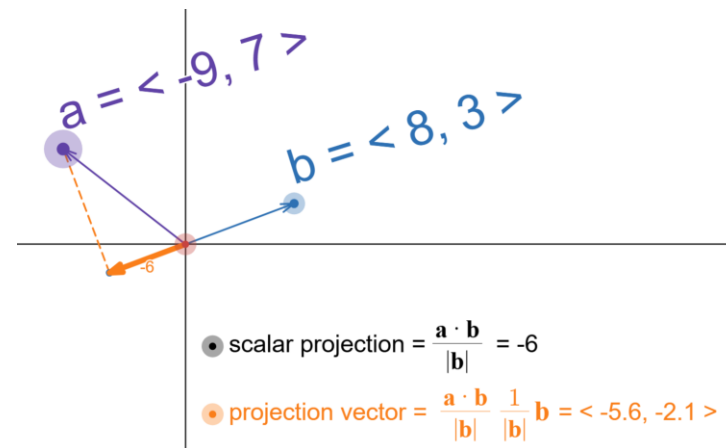
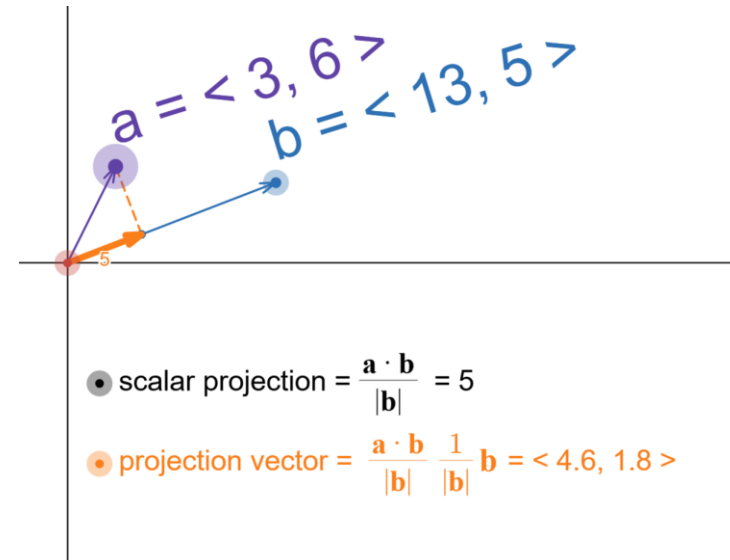
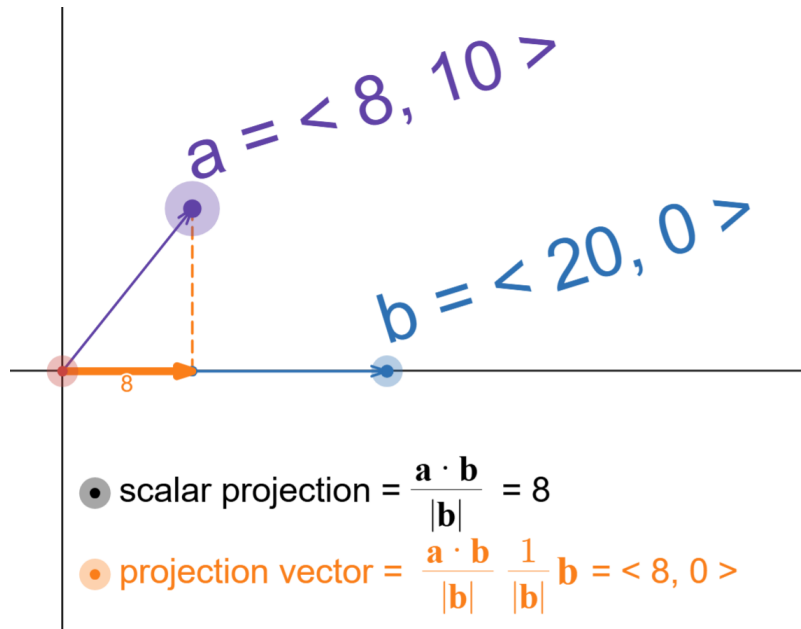
$$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$$

A NUMBER, the signed length along $\mathbf{u}$.

Vector Projection

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \left(\frac{1}{|\mathbf{u}|} \mathbf{u} \right)$$

Scaled VECTOR in direction of $\mathbf{u}$ with the length of the projection

Projections (HW #12–13)*Visual Examples**Discussion:**Can anyone think of applications of this idea?*

Summary:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

$$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \left(\frac{1}{|\mathbf{u}|} \mathbf{u} \right)$$